

Integral transforms defined by a new fractional class of analytic function in a complex Banach space

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January 14, 2016

Abstract

In this effort, we define a new class of fractional analytic functions containing functional parameters in the open unit disk. By employing this class, we introduce two types of fractional operators, differential and integral. The fractional differential operator is considered to be in the sense of Ruscheweyh differential operator, while the fractional integral operator is in the sense of Noor integral. The boundedness and compactness in a complex Banach space are discussed. Other studies are illustrated in the sequel.

Keywords: Analytic functions, Hadamard product, Fox- Wright function, norm Banach space.

1 Introduction

Fractional calculus is a major branch of analysis (real and complex) that deals with the possibility of captivating real number powers or complex number powers of operators (differential and integral). It has increased substantial admiration and significance throughout the past four decades, in line for mainly to its established requests and applications in various apparently different and extensive fields and areas of science, medicine and engineering. It does certainly deliver several possibly advantageous apparatuses for explaining and solving differential, integral and differ-integral equations, and numerous other difficulties and

problems connecting special functions of mathematical physics as well as their generalizations, modification and extensions in one and more variables (real and complex) (see [6], [11]). The utensils employed contain numerous standard and contemporary nonlinear analysis methods in real and complex, such as fixed point theory, boundedness and compactness techniques. It is beneficial to investigators and researchers, in pure and applied mathematics. The classical integral and derivative are understanding and employing with normal, ordinary and simulated methods. Fractional Calculus is a field of mathematical studies that produces out of the classical definitions of the calculus integral and derivative operators in considerable the similar technique fractional advocates is an extension of advocates with integer value.

The theory of geometric function concerns with a special class of analytic functions, which are defined in the open unit disk; such as see [10], the Koebe function of first order

$$f(z) = \frac{z}{(1-z)},$$

and of second order

$$f(z) = \frac{z}{(1-z)^2}.$$

The fractional type of analytic functions is suggested and studied in [10] as follows:

$$f(z) = \frac{z^\alpha}{(1-z)^\alpha},$$

with $\alpha = \frac{n+m}{m}$, $n, m \in \mathbb{N}$, where $\alpha = 1$, in the case $n = 1$.

In this effort, we define a new class of fractional analytic functions F in unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, with two functional parametric power as follows:

$$\begin{aligned} F(z) &= \frac{z^\mu}{(1-z^\mu)^\alpha} = z^\mu \left(\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(n)!} z^{\mu n} \right), \quad z \in \mathbb{U} \\ &= z^\mu \left(1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(n)!} z^{\mu n} \right) \\ &= z^\mu + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{\mu n} \end{aligned} \tag{1.1}$$

where $\alpha \geq 1$ and $\mu \geq 1$, not that the latter takes it value from the following relation;

$$\mu := \frac{n+m}{m} \quad m, n \in \mathbb{N}.$$

Hence, we obtain the formal of fractional analytic function:

$$F(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{\mu n},$$

where $\mu = 1$, when $n = 1$. Let \mathcal{A}_μ be the class of all analytic functions F in unit disk \mathbb{U} and take the form

$$F(z) := z + \sum_{n=2}^{\infty} \alpha_n z^{\mu n} \quad (1.2)$$

$$(\mu \geq 1, |\alpha_n| \leq \frac{(\alpha)_{n-1}}{(n-1)!}; \alpha \geq 1, n \in \mathbb{N} \setminus \{0, 1\}).$$

And let \mathcal{X}_μ be the class of all normalized analytic functions F in unit disk \mathbb{U} taking the formal

$$F(z) := z - \sum_{n=2}^{\infty} \alpha_n z^{\mu n}, \quad (1.3)$$

$$(\mu \geq 1, |\alpha_n| \leq \frac{(\alpha)_{n-1}}{(n-1)!}; \alpha \geq 1, n \in \mathbb{N} \setminus \{0, 1\}).$$

For two functions $F_j \in \mathcal{A}_\mu$, $j = 1, 2$, give by

$$F_j(z) = z + \sum_{n=2}^{\infty} \alpha_{n,j} z^{\mu n}, \quad (j = 1, 2),$$

the convolution (or Hadamard) product, denoting by $F_1 * F_2$ and taking the formal

$$F_1 * F_2(z) = z + \sum_{n=2}^{\infty} \alpha_{n,1} \alpha_{n,2} z^{\mu n},$$

And

$$(F_1 * F_2)'(z) = F_1 * F_2'(z), \quad (|z| < 1).$$

We proceed to define a new operator $\mathcal{D}^{\beta, \mu} : \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu$ by the convolution product of two functions

$$\begin{aligned} \mathcal{D}^{\beta, \mu} F(z) &= \frac{z^\mu}{(1 - z^\mu)^{\beta+1}} * F(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\beta+1)_{n-1}}{(n-1)!} \alpha_n z^{\mu n}. \end{aligned} \quad (1.4)$$

Note that

$$\mathcal{D}^{0, \mu} F(z) = F(z), \quad (z \in \mathbb{U}).$$

We expected that the operator $\mathcal{D}^{\beta, \mu}$ is closer to similarities than the Ruscheweyh differential operator (see [9]).

Next, by using (1.4) we aim to define a new integral operator denote by $\mathcal{I}_{\beta, \mu} : \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu$ as follows: define the functional

$$F_\beta = \frac{z^\mu}{(1 - z^\mu)^{\beta+1}}, \quad (z \in \mathbb{U}, \mu \geq 1, \beta \geq 1)$$

such that

$$F_\beta(z) * F_\beta^{-1}(z) = \frac{z^\mu}{1 - z^\mu}. \quad (1.5)$$

Consequently, we receive the integral operator $\mathcal{I}_{\beta,\mu}$ defined by

$$\begin{aligned} \mathcal{I}_{\beta,\mu}F(z) &= F_\beta^{-1}(z) * F(z) \\ &= \left[\frac{z^\mu}{(1 - z^\mu)^{\beta+1}} \right]^{-1} * F(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu n}. \end{aligned} \quad (1.6)$$

and it is clear that

$$\mathcal{I}_{0,\mu}F(z) = F(z), \quad |z| < 1.$$

For $\alpha \geq 1$ and $\mu \geq 1$, then the integral operator $\mathcal{I}_{1,\mu}$ is closed to the Noor Integral (see [7]) of the n -th order of function $F \in \mathcal{A}_\mu$. Corresponding to (1.7), we have the following conclusion:

$$z\mathcal{I}_{\beta,\mu}F'(z) = z + \sum_{n=2}^{\infty} \frac{\mu(n)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu n} \quad (1.7)$$

or

$$z\mathcal{I}_{\beta,\mu}F'(z) = z + \sum_{n=2}^{\infty} \mu \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta)} \alpha_n z^{\mu n}. \quad (1.8)$$

In the following section, we aim to study some properties of the integral operator $\mathcal{I}_{\beta,\mu}$.

2 Geometric properties of $\mathcal{I}_{\beta,\mu}$

In this section, we study the geometrical properties of the integral operator $\mathcal{I}_{\beta,\mu}$ of analytic functions F in class \mathcal{A}_μ . First of all, we need the following lemma, which due to Duren.

Lemma 1. (Duren [3]) Let the function $f(z) \in \mathcal{A}$ be a starlike, then $|a_n| \leq n$ for all $n \geq 2$. And if the function $f(z) \in \mathcal{A}$ is convex, then $|a_n| \leq 1$ for all $n \geq 2$.

Theorem 1. If $\alpha \geq 1$ and $\mu \geq 1$ achieving the inequality

$$\alpha(\alpha+1)\dots(\alpha+n-2) < n!$$

then $F \in \mathcal{S}_\mu^*$, and

$$|\mathcal{I}_{\beta,\mu}F(z)| \leq \Gamma(\beta+1)r^{2\mu} {}_2\Psi_1 \left[r^\mu \middle| \begin{smallmatrix} (3,1)(1,1) \\ (\beta+2,1) \end{smallmatrix} \right],$$

for all $r < 1$.

Proof. Directly by the assumption of the theorem, we obtain

$$|\alpha_n| < n$$

consequently, in view of Lemma 1, this implies that F is starlike, where

$$\alpha_1 = 1, \alpha_2 = \alpha, \alpha_3 = \frac{\alpha(\alpha+1)}{2!}, \dots, \alpha_n = \frac{\alpha(\alpha+1)\dots(\alpha+n-2)}{(n-1)!}.$$

We proceed to show that the integral operator $\mathcal{I}_{\beta,\mu}$ is bounded by a special function.

$$\begin{aligned} |\mathcal{I}_{\beta,\mu}F(z)| &= \left| z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu n} \right| \\ &\leq \Gamma(\beta+1) r^{2\mu} \sum_{n=0}^{\infty} \frac{\Gamma(n+3)\Gamma(n+1)}{\Gamma(\beta+n+2)} \frac{r^{\mu n}}{n!}, \quad |z| < r, |\alpha_n| < n \\ &= \Gamma(\beta+1) r^{2\mu} {}_2\Psi_1 \left[r^{\mu} \middle| \begin{smallmatrix} (3,1) \\ (\beta+2,1) \end{smallmatrix} \begin{smallmatrix} (1,1) \end{smallmatrix} \right], \end{aligned}$$

where ${}_2\Psi_1$ is the well known Fox-Wright function. \square

Similarly, we have the following result:

Theorem 2. If $\alpha \geq 1$ and $\mu \geq 1$ achieving the inequality

$$\alpha(\alpha+1)\dots(\alpha+n-2) < (n-1)!$$

then $F \in \mathcal{C}_{\mu}$, and

$$|\mathcal{I}_{\beta,\mu}F(z)| \leq \Gamma(\beta+1) r^{2\mu} {}_2\Psi_1 \left[r^{\mu} \middle| \begin{smallmatrix} (2,1) \\ (\beta+2,1) \end{smallmatrix} \begin{smallmatrix} (1,1) \end{smallmatrix} \right]$$

for all $r < 1$.

Proof. By the hypotheses of the theorem, we obtain

$$|\alpha_n| < 1$$

consequently, in view of Lemma 1, this yields F is convex. We proceed to show that the integral operator $\mathcal{I}_{\beta,\mu}$ is bounded by a special function.

$$\begin{aligned} |\mathcal{I}_{\beta,\mu}F(z)| &= \left| z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\beta+1)_{n-1}} \alpha_n z^{\mu n} \right| \\ &\leq \Gamma(\beta+1) r^{2\mu} \sum_{n=0}^{\infty} \frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(\beta+n+2)} \frac{r^{\mu n}}{n!}, \quad |z| < r, |\alpha_n| < n \\ &= \Gamma(\beta+1) r^{2\mu} {}_2\Psi_1 \left[r^{\mu} \middle| \begin{smallmatrix} (2,1) \\ (\beta+2,1) \end{smallmatrix} \begin{smallmatrix} (1,1) \end{smallmatrix} \right], \quad r < 1. \end{aligned}$$

\square

3 Class of uniformly convex functions

Let E be a Banach space and E^\dagger its dual. For any $A \in E^\dagger$, we interest the set $\mathcal{W}(A) := \{w \in E : A(w) \neq 0\}$ and let the set $\gamma(A) := \{w \in E : E \setminus \mathcal{W}(A)\}$. If $A \neq 0$ then $\mathcal{W}(A)$ is dense in E and $\mathcal{W}(A) \cap \hat{\mathcal{B}}$ is dense in $\hat{\mathcal{B}}$, where $\hat{\mathcal{B}} := \{w \in E : \|w\| = 1\}$. Let define \mathcal{B} be a complex Banach space and $\mathcal{H}(\mathcal{B}, \mathbb{C})$ be a family of all functions $f : \mathcal{B} \rightarrow \mathbb{C}$, such that $f(w)|_{w=0} = 0$, this means that these functions are holomorphic in \mathcal{B} and have the Fréchet derivative $f'(w)$ for all points $w \in \mathcal{B}$.

Recall that : Let Υ and Ξ be two Banach spaces, and $\Omega \subset \Upsilon$ be an open subset of V . A function $\phi : \Omega \rightarrow \Xi$ is called Fréchet differentiable at $x \in \Omega$ if there exists a bounded linear operator $\Lambda : \Upsilon \rightarrow \Xi$ such that

$$\lim_{h \rightarrow 0} \frac{\|\phi(x+h) - \phi(x) - \Lambda h\|_\Xi}{\|h\|_\Upsilon} = 0.$$

If $f \in \mathcal{H}(\mathcal{B}, \mathbb{C})$, then the function f is defined as the form:

$$f(w) = \sum_{n=1}^{\infty} \mathcal{P}_n(w). \quad (3.1)$$

Remark 1. We note that, the series $\mathcal{P}_n : E \rightarrow \mathbb{C}$ are

- 1- Uniformly convergent on some neighborhood V of the origin.
- 2- Continuous and homogeneous polynomials of degree n .

In unit disk \mathbb{U} , let denote the family CV of functions by

$$F(z) = z + \sum_{n=2}^{\infty} \alpha_n z^{\mu_n}, \quad (z \in \mathbb{U}), \quad (3.2)$$

which are convex in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

In [2], Goodman considered geometrically defined the class UCV of uniformly convex functions, which is subclass of the class CV of convex functions in \mathbb{U} . A function $f \in CV$, if normalized by $f(0) = f'(0) - 1 = 0$ and has the property that for every (positive oriented) circular arc γ contained in \mathbb{U} , with center ζ also in \mathbb{U} the image arc $f(\gamma)$ is a convex arc.

Theorem A.(Goodman [2]) Let the function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ analytic in class UCV if and only if

$$\Re \left\{ (z - \zeta) \frac{f''(z)}{f'(z)} + 1 \right\} \geq 0, \quad (z, \zeta) \in \mathbb{U} \times \mathbb{U}.$$

Lemma 2. (Goodman [2]) If $f \in \mathcal{UCV}$, then

$$|a_n| \leq \frac{1}{n}, \quad n \geq 2.$$

The following result is due to Rønning in [8] and Ma and Minda (see [5]).

Theorem B. Let the function f analytic and belongs in \mathcal{UCV} if and only if

$$\Re \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right|, \quad (|z| < 1).$$

3.1 The class of $\mathcal{UCV}_{\mathcal{A}_\mu}$ uniformly convex function

Let $A \in E^*$, $A \neq 0$. For any $f \in \mathcal{H}(\mathcal{B}, \mathbb{C})$ of the form

$$F(w) = A(w) + \sum_{n=2}^{\infty} \mathcal{P}_n(w), \quad w \in \mathcal{B} \quad (3.3)$$

for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ we put

$$F_a(z) = \frac{F(za)}{A(a)}, \quad \mu \geq 1, z \in \mathbb{U}. \quad (3.4)$$

It is clear that

$$F_a(z) = z + \sum_{n=2}^{\infty} \frac{P_n(a)}{A(a)} z^{\mu n}, \quad |z| < 1 \quad (3.5)$$

In additional, it is easy to obtain

$$F_a^{(n)}(z) = \frac{F_a^{(n)}(za)(a, \dots, a)}{A(a)}, \quad n \in \mathbb{N}, |z| < 1. \quad (3.6)$$

Let $\mathcal{UCV}_{\mathcal{A}_\mu}$ denote the family of all functions $F \in \mathcal{H}(\mathcal{B}, \mathbb{C})$ of the form (3.3) such that, for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ the function F_a belongs to the class \mathcal{UCV} . From the following results, we investigate some properties of the functions F in the class \mathcal{UCV} .

Theorem 3. (Bounded coefficient) If the function F is belong in $\mathcal{UCV}_{\mathcal{A}_\mu}$ and $a \in \hat{\mathcal{B}}$. Then

$$|\mathcal{P}_n(a)| \leq \frac{1}{n} |A(a)|, \quad n \geq 2$$

Proof. Assume that, the function $F \in \mathcal{UCV}_{\mathcal{A}_\mu}$, if $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, then $F_a \in \mathcal{UCV}$ and from here we have (3.7). In another side, if $a \in \gamma(A) \cap \hat{\mathcal{B}}$, clearly that $a = \lim_{m \rightarrow \infty} a_m$, where $a_m \in \mathcal{W}(A)$, $m \in \mathbb{N}$. There exists $r_m \in \mathbb{R}^+$ such that

$\frac{a_m}{r_m} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}, m \in \mathbb{N}$, it is clear that $(r_m, m > 0)$ is bounded for the origin is an interior point of \mathcal{B} . For $\frac{a_m}{r_m} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}, m \in \mathbb{N}$, we obtain

$$|\mathcal{P}_n(\frac{a_m}{r_m})| \leq \frac{1}{n} |A(\frac{a_m}{r_m})|, \quad n \geq 2$$

consequence

$$|\mathcal{P}_n(a_m)| \leq \frac{r_m^{n-1}}{n} |A(a_m)|, \quad n \geq 2$$

lastly by letting $m \rightarrow \infty$, we get $\mathcal{P}_n(a) = 0$. \square

Corollary 1. All the functions F which belong in $\mathcal{UCV}_{\mathcal{A}_\mu}$ are vanish on $\gamma(A) \cap \mathcal{B}$.

Corollary 2. If $F \in \mathcal{UCV}_{\mathcal{A}_\mu}$, then

$$\|\mathcal{P}_n\| \leq \frac{1}{n} \|A\|, \quad n \geq 2.$$

Theorem 4. (Sufficient condition) If the function F belongs to the class $\mathcal{UCV}_{\mathcal{A}_\mu}$ and $F'(w) \neq 0$, for all $w \in \mathcal{B}$. Then

$$\Re \left\{ 1 + \frac{F''(w)(w, w)}{F'(w)(w)} \right\} \geq \left| \frac{F''(w)(w, w)}{F'(w)(w)} \right|, \quad w \in \mathcal{W}(A) \cap \mathcal{B}. \quad (3.7)$$

Proof. Let $w \in \mathcal{W}(A) \cap \mathcal{B}$, $w \neq 0$. Then $a = \frac{w}{\|w\|} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ and thus the $F_a \in \mathcal{UCV}$. By using Theorem B, we get

$$\Re \left\{ 1 + \frac{zF_a''(z)}{F_a'(z)} \right\} \geq \left| \frac{zF_a''(z)}{F_a'(z)} \right|, \quad z \in \mathbb{U}. \quad (3.8)$$

then, by recall the equality

$$\frac{zF_a''(z)}{F_a'(z)} = \frac{F''(za)(za, za)}{F'(za)(za)}$$

then, we have

$$\left| \frac{zF_a''(z)}{F_a'(z)} \right| = \left| \frac{zF''(za)(za, za)}{F'(za)(za)} \right| \leq \left| 1 + \frac{F''(za)(za, za)}{F'(za)(za)} \right|$$

By putting $za = \|w\|$, we obtain (3.7). \square

Corollary 3. For $F \in \mathcal{H}(\mathcal{B}, \mathcal{C})$, $F'(w)|_{w=0} = A$ and $F'(w) \neq 0$, for all $w \in \mathcal{B}$. If

$$\Re \left\{ 1 + \frac{zF''(w)(w, w)}{F'(w)(w)} \right\} \geq \left| \frac{zF''(w)(w, w)}{F'(w)(w)} \right|, \quad w \in \mathcal{W}(A) \cap \mathcal{B}. \quad (3.9)$$

Then $F \in \mathcal{UCV}_{\mathcal{A}_\mu}$

Proof. Let $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$. Then $F'_a(z) = F'(za)(a) \neq 0$, $|z| < 1$ and

$$\frac{zF''_a(z)}{F'_a(z)} = \frac{F''(za)(za, za)}{F'(za)}, \quad |z| < 1.$$

□

From (3.9), we get $F_a \in \mathbb{UCV}$, for all $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, hence $F \in \mathbb{UCV}_{\mathcal{A}_\mu}$.

4 Quasi Hadamard-product

In this section, we set up some certain results which dealing with the quasi-Hadamard product of functions $F(w)$ defined by (3.3) in the class $\mathbb{UCV}_{\mathcal{A}_\mu}$. Let define

$$F_j(w) = A(w) + \sum_{n=2}^{\infty} \mathcal{P}_{n,j}(w) z^{\mu n} \quad j = \{1, 2, \dots, l\}. \quad (4.1)$$

with

$$F_{a_j}(z) = z + \sum_{n=2}^{\infty} \frac{\mathcal{P}_n(a_j)}{A(a_j)} z^{\mu n} \quad j = \{1, 2, \dots, l\}, z \in \mathbb{U}. \quad (4.2)$$

Also, let define the quasi-Hadamard product of two functions $F(w)$ and $G(w)$ in class $\mathbb{UCV}_{\mathcal{A}_\mu}$ by

$$F(w) * G(w) := A(w) + \sum_{n=2}^{\infty} \mathcal{P}(w) \Phi(w),$$

where $G(w) := A(w) + \sum_{n=2}^{\infty} \Phi(w) \quad w \in \mathcal{B}$.

Theorem 5. Let the function F_j given by (4.1) be in the class $\mathbb{UCV}_{\mathcal{A}_\mu}$ for every $j = 1, 2, \dots, l$; and let the function G_i defined by

$$G_i(w) = A(w) + \sum_{n=2}^{\infty} \Phi_{n,i}(w) z^{\mu n}, i = 1, 2, \dots, s.$$

Then the quasi Hadamard product of more two functions $F_1 * F_2 * \dots * F_l * G_1 * G_2, \dots * G_s(z)$ belongs to the class $\mathbb{UCV}_{\mathcal{A}_\mu}^{l+s}$.

Proof. Let

$$H(w) = A(w) + \sum_{n=2}^{\infty} \left\{ \prod_{j=1}^l \mathcal{P}_{n,j}(w) \prod_{i=1}^s \Phi_{n,i}(w) \right\} z^{\mu n}.$$

We aim to show that

$$\sum_{n=2}^{\infty} n^{l+s} \left\{ \prod_{j=1}^l \mathcal{P}_n(a_j) \prod_{i=1}^s \Phi_n(a_i) \right\} \leq \prod_{j=1}^l A(a_j) \prod_{i=1}^s A(a_i).$$

Since $F_j \in \mathbb{UCV}_{\mathcal{A}_\mu}$, then from Theorem 3, we obtain (4.3) and (4.4)

$$\sum_{n=2}^{\infty} n \mathcal{P}_n(a_j) \leq A(a_j)$$

for every $j = 1, 2, \dots, l$. then we have

$$\mathcal{P}_n(a_j) \leq \frac{A(a_j)}{n} \quad (4.3)$$

for every $j = 1, 2, \dots, s$. In a similar way, for $G_i \in \mathbb{UCV}_{\mathcal{A}_\mu}$ we get

$$\sum_{n=2}^{\infty} n \Phi_n(a_i) \leq A(a_i).$$

Therefore

$$\Phi_n(a_i) \leq \frac{A(a_i)}{n} \quad (4.4)$$

for every $i = 1, 2, \dots, s$. By (4.3) and (4.4), for $j = 1, 2, \dots, l$ and $i = 1, 2, \dots, s$, we attain

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[n^{l+s} \left\{ \prod_{j=1}^l \mathcal{P}_n(a_j) \prod_{i=1}^s \Phi_n(a_i) \right\} \right] \\ & \leq \left[n^{l+s} \left\{ n^{-s} n^{-l} \prod_{j=1}^l A(a_j) \prod_{i=1}^s A(a_i) \right\} \right], \\ & \leq \left\{ \prod_{j=1}^l A(a_j) \prod_{i=1}^s A(a_i) \right\}. \end{aligned}$$

Hence $H(w) \in \mathbb{UCV}_{\mathcal{A}_\mu}^{l+s}$ □

Corollary 4. Let the function $F_j(w) = A(w) + \sum_{n=2}^{\infty} \mathcal{P}_{n,j}(w) z^{\mu n}$ given by (4.1) be in the class $\mathbb{UCV}_{\mathcal{A}_\mu}$ for every $j = 1, 2, \dots, l$. Then the Hadamard product $F_1 * F_2, \dots, F_l(z)$ belongs to the class $\mathbb{UCV}_{\mathcal{A}_\mu}^l$.

Corollary 5. Let the function $G_i(w) = A(w) + \sum_{n=2}^{\infty} \Phi_{n,i}(w) z^{\mu n}$, defined by (4.1) be in the class $\mathbb{UCV}_{\mathcal{A}_\mu}$ for every $i = 1, 2, \dots, s$. Then the Hadamard product $G_1 * G_2, \dots, G_s(z)$ belongs to the class $\mathbb{UCV}_{\mathcal{A}_\mu}^s$.

5 Applications

In this section, we introduce some applications dealing with a complex Banach space.

Campbell in [9] studied a complex norm Banach space structure of the class of locally univalent functions of finite order, where the order of a function $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is known as

$$\|f\|_T = \sup_{z \in \mathbb{U}} (1 - |z|^2) |T_f| < \infty, \quad (|z| < 1).$$

where $T_f = f''/f'$ denotes by the pre-Schwarzian derivative of function f .

Theorem 6. (Boundedness) Let $F \in \mathcal{UCV}_{\mathcal{A}_\mu}$. Then

$$|\mathcal{I}_{\beta,\mu} F| \leq 1 + M \|F(w)\|_T, \quad w \in \mathcal{W}(A) \cap \mathcal{B}$$

where $M = (1/(1 - |z|^2))$.

Proof. Supposing $F(w) \in \mathcal{UCV}$ such that $A(w) \neq 0, w \in \mathcal{B}$ and for any $\|a\| = 1, a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, we get

$$\begin{aligned} \left| \frac{z \mathcal{I}_{\beta,\mu} F_a''(z)}{\mathcal{I}_{\alpha,\mu} F_a'(z)} \right| &= \left| \frac{F_\beta^{-1}(z) * z F_a''(z)}{F_\beta^{-1}(z) * F_a'(z)} \right| \\ &= \frac{(1 - |z|^2) |z F_a''(z) / F_a'(z)|}{(1 - |z|^2)} \\ &\leq (1 - |z|^2) \left| 1 + \frac{z F_a''(z)}{F_a'(z)} \right| \frac{1}{(1 - |z|^2)} \\ &= (1 - |z|^2) \left| 1 + \frac{F''(za)(za, za)}{F'(za)(za)} \right| \frac{1}{(1 - |z|^2)} \end{aligned}$$

In view of Theorem 4, we lead to

$$\left| \frac{z \mathcal{I}_{\beta,\mu} F_a''(z)}{\mathcal{I}_{\beta,\mu} F_a'(z)} \right| \leq 1 + \frac{\|F(za)\|_T}{(1 - |z|^2)}. \quad (5.1)$$

By setting the supremum for the last assertion over the unit disk \mathbb{U} and putting $z = \|w\|$, the boundedness of the operator $\mathcal{I}_{\beta,\mu} F_a(z)$ is satisfied. \square

Theorem 7. (Compactness) For $F \in \mathcal{UCV}_{\mathcal{A}_\mu}$, then the integral operator $\mathcal{I}_{\beta,\mu} F$ is compact in complex norm Banach space.

Proof. If $\mathcal{I}_{\beta,\mu} F_a$ is compact, then the function F_a is bounded and by Theorem 6, it is follow that $F \in \mathcal{B}$ the integral operator $\mathcal{I}_{\beta,\mu}$ is compact. Let suppose that $\mathcal{I}_{\beta,\mu} \in \mathcal{UCV}_{\mathcal{A}_\mu}$, that $F_{a_m}, m \in \mathbb{N}$ is a sequence in Banach space, and $F_{a_m} \rightarrow 0$ uniformly on $\bar{\mathbb{U}}$ as $m \rightarrow \infty$. For every $\varepsilon > 0$, there is $\delta \in (0, 1)$ such that

$$\frac{1}{(1 - |z|^2)} < \varepsilon$$

where $\delta < |z| < 1$, since δ is arbitrary. then

$$\begin{aligned}
\left| \frac{z \mathcal{I}_{\beta, \mu} F''_{a_m}(z)}{\mathcal{I}_{\beta, \mu} F'_{a_m}(z)} \right| &= \sup_{z \in \mathbb{U}} \left| \frac{F_{\beta}^{-1}(z) * z F''_{a_m}(z)}{F_{\beta}^{-1}(z) * F'_{a_m}(z)} \right| \\
&= \sup_{z \in \mathbb{U}} \left\{ \frac{(1 - |z|^2) |z F''_{a_m}(z) / F'_{a_m}(z)|}{(1 - |z|^2)} \right\} \\
&\leq 1 + \sup_{z \in \mathbb{U}} \left\{ (1 - |z|^2) \left| \frac{F''_m(za)(za, za)}{F'_m(za)(za)} \right| \frac{1}{(1 - |z|^2)} \right\} \\
&\leq 1 + \varepsilon \|F_m(w)\|_T.
\end{aligned} \tag{5.2}$$

Since for $F_m \rightarrow 0$ on $\overline{\mathbb{U}}$ we get $\|F_m\|_T \rightarrow 0$, and that ε is a arbitrary number, by setting $m \rightarrow \infty$ in (5.2), we have that

$$\lim_{m \rightarrow \infty} \|\mathcal{I}_{\beta, \mu} F_{a_m}\|_T = 0.$$

Therefore, $\mathcal{I}_{\beta, \mu}$ is compact. \square

Theorem 8. If $F \in \mathbb{UCV}_{\mathcal{A}_\mu}$, then for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, we have

$$\left| |\mathcal{I}_{\beta, \mu} \{F(w)\}| - |A(w)| \right| \leq \frac{\alpha A(a)}{2\beta} |z|^{2\mu} \quad \beta \geq 1, \alpha \geq 1. \tag{5.3}$$

Proof. Let $F \in \mathbb{UCV}_{\mathcal{A}_\mu}$, then we have

$$\sum_{n=2}^{\infty} \mathcal{P}_n(a) \leq \frac{A(a)}{n}, \quad a \in \mathcal{W}(A) \cap \hat{\mathcal{B}} \tag{5.4}$$

From (3.5), (3.7) and (1.6), we obtain

$$\begin{aligned}
\mathcal{I}_{\beta, \mu} F_a(z) &= F_{\beta}^{-1} * F_a(z) \\
&= z + \sum_{n=2}^{\infty} \psi(n) \mathcal{P}_n(a) z^{\mu n},
\end{aligned} \tag{5.5}$$

where $\psi(n) = \frac{(\alpha+1)_{n-1} A(a)}{n(\beta+1)_{n-1}}$.

We note that the function $\psi(n)$ is a non-increasing function for integral $n \in \mathbb{N} \setminus \{0, 1\}$. Therefore, we obtain

$$\psi(n) \leq \psi(2) = \frac{\alpha A(a)}{2\beta} \quad \alpha \geq 1, \beta \geq 1.$$

Using (5.4) and (5.5), hence the assertion (5.3) of Theorem 8 is easily arrived at. \square

6 Conclusion

We generalized a class of analytic functions (Koebe type), by utilizing the concept of fractional calculus. This class involves the well known geometric functions in the open unit disk. Moreover, by employing the above class, we defined two types of fractional operators, differential and integral. The fractional differential operator is supposed to be in the sense of Ruscheweyh differential operator, while the fractional integral operator is assumed to be in the sense of Noor integral. Some geometrical properties are illustrated for the integral operator such as the starlikeness and convexity. Topological properties are investigated in a complex Banach space, such as the boundedness and compactness. The unusual product (Hadamard-Owa product) is discussed in some classes that involving the fractional integral operator.

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